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Robust Continuous Finite Time Control of Double-Integrator Systems with Unknown Control Direction

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Abstract—**A continuous sliding mode controller using super-twisting algorithm is presented for a class of double integrator systems with constant unknown control direction. The system is also considered to be perturbed by unknown non-vanishing Lipschitz disturbances. In contrast to existing continuous control solutions on this subject that at best can achieve exponential stability, the developed controller yields finite time convergence of the states to the origin of the system. Simulation results are provided to demonstrate the robustness of the controller to unknown control direction.**

I. Introduction

Unknown control direction refers to a fixed or time-varying uncertainty in the sign of the input gain. For a system such as $\dot{x}(t) = f(\cdot) + bu(t)$, the sign of b governs the direction of the control input $u(t)$ applied to the system. Uncertainties in the sign of b as a result of, for instance, variations in the operating environment, manufacturing faults, or adversarial attacks can lead to performance degradation and instability of the system. The challenges in the controller design due to unknown control direction have been successfully addressed in the adaptive control framework using Nussbaum functions [1]. Variations of the Nussbaum function approach have also been developed for numerous systems [2]–[7] with uncertainty in the control direction. Although the Nussbaum function approach can adapt to the uncertain sign of the input gain, it can only guarantee asymptotic stability of the system, and it exhibits an undesirable peaking phenomenon as a result of high-gain feature of the controller [3], [8]. Apart from the Nussbaum function approach, monitoring functions [9]–[11] and minimum seeking Lyapunov function [12] have also been developed to compensate for the input sign uncertainty. While the aforementioned methods are able to address the issue of unknown control direction, most results lack stronger, beyond asymptotic, stability guarantees and cannot accommodate unknown nonlinearities in system dynamics (e.g., arising due to exogenous disturbances).

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Robust control theory, and, in particular, sliding mode control (SMC) [13], provides another avenue to compensate for the unknown control direction along with its inherent capacity for disturbance rejection. In [8], Bartolini *et al.* developed a SMC-based switching controller that yielded asymptotic stability for systems with constant unknown control direction and uncertain drift dynamics. In [14], Drakunov *et al.* compensated for the input sign uncertainty by designing a periodic switching function that slides along multiple-equilibrium surfaces. Our previous work [15] extended the concept in [14] to develop a nonsingular terminal SMC that achieves finite time convergence of the states to the origin of the system in the presence of constant input sign uncertainty and unknown state-varying disturbances. Although discontinuous controllers can yield improved stability, often the concern is about large bandwidth actuation and chattering that results from high frequency switching over the sliding surface.

Various control techniques have been developed to attenuate chattering by the introduction of, for example, boundary layer [16], observers [17], multi-phase converters and phase shift [18], low-pass filter [19], and gain adaptation [20]–[27]. In [28], we presented an adaptive terminal SMC method for a second-order system with time-varying unknown control direction, where the control gain is varied to alleviate chattering and maintain sliding motion with reduced control effort. Higher order sliding mode (HOSM) control can also be used to mitigate chattering. A well-known subclass of HOSM is the super-twisting algorithm (STA), which, unlike HOSM, does not require the knowledge of the derivatives of the sliding variable. Due to this advantage, variations of the STA have been developed in [29]–[31]. In [32], we developed an STA-based controller that yields exponential stability in the presence of constant unknown control direction. However, recent results in homogeneous sliding mode control [31], [33], [34] are able to guarantee finite time convergence of the system to the origin. Motivated by the results in [33], it is of our interest to extend the STA in [32] to achieve finite time convergence for systems with unknown control direction.

The contribution of the presented work is in the development of an STA-based ***continuous controller*** with ***finite time stability*** for systems with constant unknown control direction. We consider systems with double integrator dynamics and relative degree one. The magnitude of the scalar input gain is assumed to be known, but its sign is unknown. The developed controller does not implement logic tests [8], [35] or monitoring functions [9], [10] to determine the input sign uncertainty. As opposed to [8]–[10], [14], [15], [28], [35], the control input presented in this paper is continuous in time. Also, in constrast to [7], [32], [36]–[38], the controller guarantees finite time convergence of the system to the origin. Additionally, the developed controller can compensate for unknown non-vanishing disturbances unlike Nussbaum function based adaptive framework. The robustness of the developed STA-based controller to input sign uncertainty is verified through numerical simulations results. Comparison to prior work: Although finite time convergence of the states to the origin in guaranteed in [15] and [28], the terminal SMC-based controllers therein are discontinuous in nature. The discontinuity was removed by developing an STA-based controller in [32], which ensures exponential stability of the system. This paper presents a non-trivial extension of [32] by considering a new sliding surface and redesigning the controller to achieve finite time convergence of the states to the origin.

II. Problem Formulation

Consider an uncertain system with double integrator dynamics subjected to non-vanishing disturbance as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(t) + bu \end{aligned} \tag{1}$$

where $x = [x_1 \; x_2]^T \in \mathbb{R}^2$ is the state of the system, $u(t) \in \mathbb{R}$ is the control input, $b \in \mathbb{R}$ denotes the constant input gain, and $f(t) \in \mathbb{R}$ represents the unknown non-vanishing disturbance.

Assumption 1: The function $f(t)$ is Lipschitz continuous with a known Lipschitz constant, i.e., the time-derivative $\dot{f}(t) \in \mathbb{R}$ can be upper bounded by a known constant $\bar{\dot{f}} \in \mathbb{R}^+$ as

$$|\dot{f}(t)| \leq \bar{\dot{f}}.$$

Assumption 2: The constant b in (1) can be written as $b = |b| sgn\,(b)$, which satisfies the following properties:

- The magnitude of b is known and $|b| > 0$.
- The sign of b is constant but unknown, and since $|b| > 0$, $sgn(b) \in \{1, -1\}$.

Assumption 3: The system in (1) is controllable.

The objective is to design a continuous robust control input $u(t)$ that guarantees finite time stability of the system in (1) in the presence of the input sign uncertainty and non-vanishing disturbances.

III. Sliding Surface Design

Based on the SMC theory, to ensure that (1) converges to the origin in finite time, the sliding surface $s(t)$ can be designed as

$$s = x_2 + \int_0^t \alpha_1 \lfloor x_1 \rceil^{p_1} + \alpha_2 \lfloor x_2 \rceil^{p_2} \, d\tau \tag{2}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}^+$ are constants, $p_1 \in (0, 1)$, $p_2 = \frac{2p_1}{1+p_1}$, $\lfloor x_1 \rceil^{p_1} = |x_1|^{p_1} sgn(x_1)$, and $\lfloor x_2 \rceil^{p_2} = |x_2|^{p_2} sgn(x_2)$. When the sliding surface $\dot{s}(t) = 0 : \dot{x}_2 = -\alpha_1 \lfloor x_1 \rceil^{p_1} - \alpha_2 \lfloor x_2 \rceil^{p_2}$, and $x(t)$ approaches the origin in finite time [33]. To compensate for the sign uncertainty in b, the hypersurface $\tilde{s}(t)$ can be designed as

$$\tilde{s} = s + \lambda \int_0^t \lfloor s \rceil^{p_3} \, d\tau \tag{3}$$

where $\lambda \in \mathbb{R}^+$ is a constant, and $p_3 \in (0, 1)$.

Taking time derivative of (2) and (3), we obtain

$$\dot{s} = \dot{x}_2 + \alpha_1 \lfloor x_1 \rceil^{p_1} + \alpha_2 \lfloor x_2 \rceil^{p_2} \tag{4}$$

$$\dot{\tilde{s}} = \dot{s} + \lambda \lfloor s \rceil^{p_3}. \tag{5}$$

Substituting (1) and (4) into (5), the open-loop system can be obtained as

$$\dot{\tilde{s}} = bu + f(t) + s_1 \tag{6}$$

$$s_1 = \alpha_1 \lfloor x_1 \rceil^{p_1} + \alpha_2 \lfloor x_2 \rceil^{p_2} + \lambda \lfloor s \rceil^{p_3}. \tag{7}$$

The hypersurface $\tilde{s}(t)$ is used in the subsequent analysis to compensate for the unknown control direction. It will be shown that $\tilde{s}(t)$ approaches a constant in finite time, hence $s(t)$ decays to zero in finite time, and thus the state $x(t)$ reaches the origin in finite time.

IV. Controller Development

The control input $u(t)$ in (1) is designed below. To facilitate subsequent analysis, $u(t)$ is segregated into two terms as

$$\begin{aligned} u &= |b|^{-1} \left(k_1 \lfloor \Psi \rceil^{1/2} + \int_0^t u_2(\tau) \, d\tau + \sigma_L(\Omega) \, s_1 \right) \\ u_2 &= k_2 sgn\,(\Psi) \end{aligned} \tag{8}$$

where $k_1, k_2 \in \mathbb{R}^+$ are constants to be defined later, $\lfloor \Psi \rceil^{1/2} = |\Psi|^{1/2} \, sgn(\Psi)$, $\sigma_L(\Omega) = \sigma\left(\frac{\Omega}{L}\right)$ is the saturation function with maximum magnitude of 1 and linear range $L < 1$, s_1 is defined in (7), and $\Psi(t)$ and $\Omega(t)$ are sinusoidal function of the hyper sliding surface $\tilde{s}(t)$ defined as

$$\Psi(t) \triangleq \sin \frac{\pi \tilde{s}}{\varepsilon}, \quad \Omega(t) \triangleq \cos \frac{\pi \tilde{s}}{\varepsilon}$$

where $\varepsilon \in \mathbb{R}^+$ is a constant that determines the spacing between the equilibrium surfaces.

Using the super-twisting algorithm in [29] to represent the surface $s(t)$ as a second order system, the hypersurface $\tilde{s}(t)$ can also be written in the same manner. Substituting (8) into the open-loop system in (6), the closed-loop system can be obtained as

$$\begin{aligned} \dot{\tilde{s}} &= b_1 k_1 \lfloor \Psi \rceil^{1/2} + z + (b_1 \sigma_L(\Omega) + 1) s_1 \\ \dot{z} &= b_1 k_2 sgn\,(\Psi) + \dot{f}(t) \end{aligned} \tag{9}$$

where $b_1 = sgn(b)$. The constant k_2 in (9) is designed to satisfy the inequality $k_2 > \bar{\dot{f}}$.

To facilitate the subsequent analysis, the term $\dot{z}(t)$ can be succinctly written as

$$\dot{z} = \frac{\lfloor \Psi \rceil^{1/2}}{2 |\Psi|^{1/2}} (2\rho(t))$$

where the function $\rho(t) \in \mathbb{R}$ is defined as

$$\rho \triangleq b_1 k_2 + \dot{f}(t) sgn(\Psi). \quad (10)$$

Following similar procedures as in [29] and [32], let the vector $\zeta(t) \in \mathbb{R}^2$ be defined as

$$\zeta(t) = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} \triangleq \begin{bmatrix} \lfloor \Psi \rceil^{1/2} \\ z \end{bmatrix}. \quad (11)$$

Taking time derivative of $\zeta(t)$ along the trajectory in (5), $\dot{\zeta}(t)$ can be obtained as

$$\dot{\zeta} = \frac{1}{2|\Psi|^{1/2}} (A\zeta + B). \quad (12)$$

In (12), the time- and state-varying matrix $A(x,t)$ and state-varying vector $B(x)$ can be obtained as

$$A = \begin{bmatrix} \frac{\pi \Omega b_1 k_1}{\varepsilon} & \Omega \frac{\pi}{\varepsilon} \\ 2\rho & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{\pi}{\varepsilon}\Omega g \\ 0 \end{bmatrix} \quad (13)$$

where $g = (b_1 \sigma_L(\Omega) + 1) s_1$. The expression in (12) is obtained using the fact that the distributional product of the delta function $\delta(x)$ with x is zero, i.e., $x\delta(x) = 0$.

Theorem 1: For the system in (1), where the sign of the input gain b is constant and unknown, provided that k_1, k_2 and ε are designed to satisfy inequalities

$$k_1 + \mu_1 \mu_2^{-1} > \mu_3 \mu_2^{-1} \qquad \frac{2\pi|\Omega|}{\varepsilon} > \frac{1}{2} \quad (14)$$

$$|\rho| > \left| \frac{\pi}{\varepsilon} \Omega \gamma k_1 \right| \qquad k_2 \geq \bar{\dot{f}} \quad (15)$$

where

$$\mu_1 = |\gamma| k_1 + 2(\beta + \gamma^2)\mu_4 + \frac{2\pi|\Omega\rho|}{\varepsilon}(\beta + \gamma^2) \quad (16)$$

$$\mu_2 = 2(\beta + \gamma^2)\frac{\pi^2|\Omega\gamma|}{\varepsilon^2}\left(2|\Omega\gamma_s| - \frac{\varepsilon}{2\pi}\right) + 3\frac{\pi}{\varepsilon}|\Omega\gamma\rho| \quad (17)$$

$$\mu_3 = \left(\frac{\pi\Omega}{\varepsilon}\right)^2 (\beta + \gamma^2) + 4\rho^2 + \left(4|\gamma\rho| + \frac{|\gamma|}{2}\right)\frac{\pi|\gamma|}{2}\left(2|\Omega\gamma_s| - \frac{\varepsilon}{2\pi}\right) \quad (18)$$

$$\mu_4 = \frac{\pi|\Omega\rho|}{\varepsilon} - \left(\frac{\pi\Omega}{\varepsilon}\right)^2 |\gamma| k_1 \quad (19)$$

$$\gamma_s = sgn(\gamma) \quad (20)$$

then the control input $u(t)$ in (8) ensures that the surface $\tilde{s}(t)$ is reached in finite time, and (s, x) is finite time stable.

Proof: Consider a Lyapunov candidate function $V(t) \in \mathbb{R}^+$ as

$$V = \zeta^T P \zeta \quad (21)$$

where $P \in \mathbb{R}^{2\times2}$ is a positive definite symmetric matrix given by

$$P = \begin{bmatrix} \beta + \gamma^2 & \gamma \\ \gamma & 1 \end{bmatrix} \quad (22)$$

where $\gamma = -k_\gamma sgn(\Omega)$ and $\beta, k_\gamma \in \mathbb{R}^+$ are constants. Taking time derivative of (21) along the trajectories of (1) and substituting (12), the Lyapunov derivative can be obtained as

$$\dot{V} = \frac{1}{2|\Psi|^{1/2}}\left(\zeta^T\left(A^T P + PA\right)\zeta + B^T P\zeta + \zeta^T PB\right) + \zeta^T \dot{P}\zeta. \quad (23)$$

Remark 1: Consider a set where $\tilde{s} = \{\epsilon/2, 3\epsilon/2, 5\epsilon/2, \ldots\}$. In this set $\Omega = 0$, hence $\dot{\gamma} = -k_\gamma\delta$, where δ is the Dirac delta function. Although $\dot{\gamma}$ is singular, $V(t)$ does not immediately go to infinity due to the property $\int_{\gamma=-\infty}^{\gamma=\infty}\delta(\gamma)d\gamma = 1$. For example, consider the case when the states are on an unattractive manifold, i.e $\tilde{s} = \varkappa + \varepsilon$, where $\varkappa = 0, \pm\varepsilon, \pm2\varepsilon, \pm3\varepsilon, \ldots$. This implies, from (9), that the hypersurface $\tilde{s}(t)$ increases to another constant $\tilde{s} = \varkappa + 2\varepsilon$ or decreases to $\tilde{s} = \varkappa$, where both surfaces are attractive. This also implies that Ω continuously increases from -1 to 1, or decreases from 1 to -1, while crossing the boundary $\Omega = 0$. Although $\dot{\gamma} = -k_\gamma\delta$, when $\Omega = 0$, the control input $u(t)$ is continuous and the actual system in (1) does not jump. Hence, the discontinuity at $\Omega = 0$ can be regarded as a jump from an unattractive set towards an attractive set. This implies that Lyapunov candidate function in (21) is continuous when the system is stable, i.e. the correct sign of b is identified, which is equivalent to the Lyapunov function considered in [29]. It will be shown that the system is stable as $\Omega \to \pm1$, and Ω only crosses the boundary $\Omega = 0$ when it goes from an unattractive to attractive region. Also, note that $\Omega = 0$ does not correspond to the equilibrium point. Therefore, the following analysis ignores the case when $\Omega = 0$ and hence $\dot{P} = 0$ is used in (23).

The Lyapunov derivative can be upper bounded as

$$\dot{V} \leq \frac{1}{2|\Psi|^{1/2}}\left(\zeta^T Q\zeta + B^T P\zeta + \zeta^T PB\right) \quad (24)$$

where $Q = A^T P + PA$. The inequality in (24) can be written as

$$\dot{V} \leq \frac{1}{2|\Psi|^{1/2}}\zeta^T Q\zeta + \frac{1}{2|\Psi|^{1/2}}\left(B^T P\zeta + \zeta^T PB\right) \quad (25)$$

where $Q(t) \in \mathbb{R}^{2\times2}$ is obtained as follows:

$$Q = \begin{bmatrix} \upsilon & \chi \\ \chi & 2\gamma\Omega\frac{\pi}{\varepsilon} \end{bmatrix}. \quad (26)$$

The scalar functions $\chi(t), \upsilon(t) \in \mathbb{R}$ in (26) are as below

$$\upsilon = 2(\beta + \gamma^2)\Omega\frac{\pi}{\varepsilon}b_1 k_1 + 4\gamma\rho \quad (27)$$

$$\chi = \gamma\frac{\pi}{\varepsilon}\Omega b_1 k_1 + 2\rho + (\beta + \gamma^2)\frac{\pi}{\varepsilon}\Omega. \quad (28)$$

The objective is to ensure $Q(t)$ to be negative definite. Let $Q_1(t) = -Q(t)$ and consider

$$Q_2 = Q_1 - \frac{|\gamma|}{2}I \quad (29)$$

where $I \in \mathbb{R}^{2\times2}$ is the identity matrix. Then, $Q_2(t)$ can be expressed as

$$Q_2 = -\begin{bmatrix} \upsilon + |\gamma|/2 & \chi \\ \chi & \sigma \end{bmatrix} \quad (30)$$

where $\sigma(t) = 2\gamma\Omega\frac{\pi}{\varepsilon} + \frac{|\gamma|}{2}$.

Recall that the eigenvalues are related to the determinant and trace of the matrix as

$$det(Q_2) = \eta_1\eta_2, \quad trace(Q_2) = \eta_1 + \eta_2 \tag{31}$$

where η_1 and η_2 are the eigenvalues of $Q_2(t)$. When $det(Q_2) > 0$ and $trace(Q_2) > 0$, this implies that the eigenvalues $\eta_1, \eta_2 > 0$ and the matrix $Q_2(t)$ is positive definite. To ensure that $Q_2(t)$ has positive eigenvalues, using the expressions in (31) and the attractive properties of $sgn(\Psi)$, the inequalities in (14) and (15) have to be satisfied. From (29) and the definition of Q_1, the positive definiteness of Q_2 ensures that Q is negative definite.

For unknown $sgn(b)$, when the sliding surface $\tilde{s}(t) = \Xi$ is attractive, where Ξ is a constant, the sign of Ω always converges to the opposite of the sign of b, i.e. $sgn(\Omega) = -sgn(b)$. This implies that $sgn(b\Omega) = -1$ when $sgn(\Psi)$ is attractive, otherwise $sgn(b\Omega) = 1$. However, when $sgn(\Psi)$ is not attractive, the state $\tilde{s}(t)$ leaves the current manifold and enters the region where $sgn(\Psi)$ is attractive.

In the set $\mathcal{F} = \{\Omega | b_1\sigma_L(\Omega) = -1\}$, there exist a positive constant β, a function $\gamma(t)$, and positive constants k_1, k_2, ε that satisfy inequalities (14)-(15). Moreover, within the set $\mathcal{F}$ where $\tilde{s}(t) = \Xi$ is attractive, $b_1\sigma_L(\Omega) = -1$ and $g = 0$, which implies that $B^T P\zeta + \zeta^T PB = 0$. Therefore, using (29) and the fact that $Q_1(t) = -Q(t)$, the Lyapunov derivative can be upper bounded as

$$\dot{V} \leq -|\gamma|\frac{\|\zeta\|^2}{4|\Psi|^{1/2}} \tag{32}$$

where $\|\cdot\|$ denotes the Euclidean norm. From (11), it is clear that $\|\zeta\| \geq |\Psi|^{1/2}$. Using this fact and the definition of $\gamma(t)$ in (22), the inequality in (32) can be expressed as

$$\dot{V} \leq -\frac{k_\gamma}{4}\|\zeta\|. \tag{33}$$

The Lyapunov function $V(t)$ in (21) can be upper and lower bounded as

$$\Lambda_{\min}\{P\}\|\zeta\|^2 \leq V \leq \Lambda_{\max}\{P\}\|\zeta\|^2$$

where $\Lambda_{\min}\{P\}$ and $\Lambda_{\max}\{P\}$ are the minimum and maximum eigenvalues of P, respectively. Therefore, the Lyapunov derivative in (33) can be further upper bounded as

$$\dot{V} \leq -\frac{k_\gamma}{4\sqrt{\Lambda_{\max}\{P\}}}V^{1/2}. \tag{34}$$

The inequality in (34) implies that $V(t)$ goes to zero in finite time [29]. This implies that $\tilde{s}(t)$ goes to a constant in finite time, which in conjunction with (5) yields the following equality:

$$\dot{s} = -\lambda\lfloor s\rceil^{p_3}. \tag{35}$$

From (35), it is clear that $s(t)$ decays to zero in finite time. Given the fact that $s(t) \to 0$ in finite time, it can easily be shown from (2) and (4) that $x(t) \to 0$ in finite time [14], [33].

Similar to [29], the proposed control structure cannot stay on the set $\mathcal{S} = \{(\Psi, z) \in \mathbb{R}^2 | \Psi = 0\}$. Since $V(t)$ is continuously decreasing, by using the Lyapunov theorem for differential inclusions stated in Proposition 14.1 in [39], where $V(t)$ does not need to be continuously differentiable, it can be concluded that the equilibrium (Ψ, z) can be reached in finite time from any initial condition.

Remark 2: In (9), let $|g| > |b_1k_1\lfloor\Psi\rceil^{1/2} + z|$, then a few behaviors manifest themselves on the sliding surface $\tilde{s}(t)$. Since $sgn(g)$ dominates the direction of motion of $\tilde{s}(t)$ in (9), following cases can be considered to analyze the stability of the system:

Case (I): Let s_1 be sign definite, i.e. $sgn(s_1) = 1$, for a time interval $[t_0, t_1]$, and $sgn(\Omega)$ is changing. Then $sgn(\dot{\tilde{s}}) = sgn(s_1)$, which implies that $\tilde{s}(t)$ is monotonically increasing. This implies that the hypersurface eventually reaches an attractive sliding surface $\tilde{s}_\Psi(t) = k_\Psi\varepsilon$, where k_Ψ are odd/even integers depending on the sign of b, which implies that $g = 0$ and the proper sign has been identified. Similar analysis can be done when $sgn(s_1) = -1$.

Case (II): The second case is when $sgn(\Omega)$ is held constant, i.e., $sgn(\Omega)$ does not change, and $sgn(s_1)$ varies. When $sgn(s_1)$ changes on the attractive set $\tilde{s}_a = k_a\varepsilon$, where $k_a \in \mathbb{R}$ can be odd/even integers depending on the sign of b, the function $g = 0$ when $\tilde{s} = k_a\varepsilon \pm \alpha$, where $0 < \alpha < \pi/2$. This case is trivial, and the proper sign has been identified. When $sgn(s_1)$ changes on the unattractive set $\tilde{s}_u = k_u\varepsilon \pm \alpha$, where $k_u \in \mathbb{R}$ are odd/even integers depending on the sign of b, and $g \neq 0$, the function $z(t)$ in (9) increases monotonically due to the fact that $sgn(b_1k_2sgn(\Psi) + \dot{f}(t)) = sgn(b_1sgn(\Psi)) = \pm 1$ is held constant. This implies that $z(t)$ eventually dominates $g(t)$, and pushes the hyper-surface towards an attractive sliding surface $\tilde{s}_a(t) = k_a\varepsilon$.

Case (III): The third case is when $sgn(\Omega)$ and $sgn(s_1)$ are switching at the same time. In this case, $sgn(\dot{\tilde{s}}) = sgn(s_1)$ and there are two possible trajectories. First, if $s_1(t)$ changes in such a way that $\tilde{s}(t) = \beta$: (a) If $\beta = k_a\varepsilon \pm \alpha$, then the sign of b is identified; (b) If $\beta = k_u\varepsilon \pm \alpha$, then we have Case (II), and $z(t)$ eventually grows and dominate $g(t)$. Second, if $s_1(t)$ changes in such a way that $\tilde{s}(t)$ is unstable, i.e. $\tilde{s}(t)$ grows, then $\tilde{s}(t)$ eventually reaches an attractive sliding surface $\tilde{s}_\Psi(t) = k_\Psi\varepsilon$. ■

V. Simulation Results

Simulation for the second order system in (1) was carried out to validate the proposed controller. Control gains and design parameters used in the simulation are as follows:

$$k_1 = 1.0 \quad k_2 = 2 \quad \alpha_1 = 1 \quad \alpha_2 = 1$$
$$p_1 = 1/2 \quad p_2 = 2/3 \quad p_3 = 1/2 \quad \varepsilon = 10$$
$$\lambda = 1 \quad L = 0.3.$$

The non-vanishing disturbance $f(t)$ and the unknown input gain b were selected as

$$f(t) = \sin(t), \quad b = 1.$$

Although the direction of the control input, i.e., $sgn(b) = 1$, was specified, it was not used in the controller development. The initial conditions were selected as

$$x_1(0) = 20, \quad x_2(0) = 5.$$

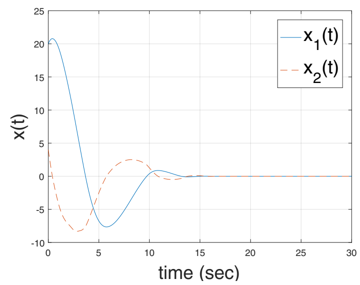

Fig. 1: Position $x_1(t)$ and velocity $x_2(t)$ versus time showing finite time convergence of the states to the origin.

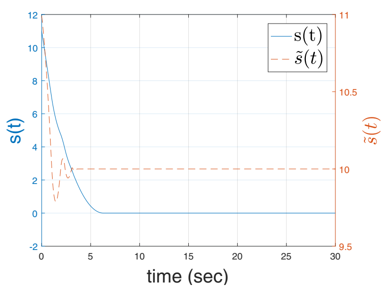

Fig. 2: Surface $s(t)$ and hypersurface $\tilde{s}(t)$ versus time showing finite time convergence of the hypersurface $\tilde{s}(t)$ and finite time convergence of the surface $s(t)$.

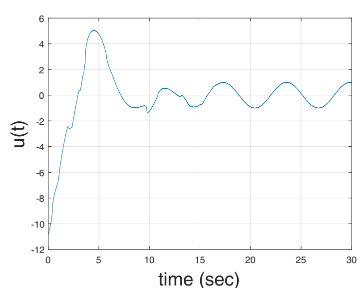

Fig. 3: Control input $u(t)$ versus time.

Fig. 1 shows the position $x_1(t)$ and the velocity $x_2(t)$, Fig. 2 shows the surface $s(t)$ and the hypersurface $\tilde{s}(t)$, and Fig. 3 shows the control input $u(t)$ as a function of time. In Fig. 2, it can be seen that $\tilde{s}(t)$ reaches a constant in finite time and consequently $s(t)$ goes to zero in finite time, which implies that $x(t)$ also goes to zero in finite time (see Fig. 1). It can be seen that the control input $u(t)$ in Fig. 3 is continuous and bounded, and does not exhibit significant chattering.

VI. Conclusions

A continuous sliding mode controller is developed for a class of second-orders systems with constant unknown control direction. The presented control structure guarantees that the sign of the input gain is identified in finite time as the hypersurface converges to a manifold. Once the control direction is identified through appropriate attractive hypersurface, the controller guarantees that the origin of the system is finite time stable. Simulation results demonstrate the robustness of the control algorithm.

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